Solution 1

We need to find the extrema of $f(x,y,z) = x^2 + 2y - z^2$ subject to $g(x,y,z) = 2x^2 + y^2 + z^2 = 4$. The Lagrange multiplier equations are $\nabla f = \lambda \nabla g$ and $g = 4$:

(4p)

$$\begin{aligned} 2x = 4\lambda x &\implies 2x(1-2\lambda) = 0 \quad \text{alt. key eq.} & (1)\\ 2 = 2\lambda y &\implies \lambda y = 1 & (2)\\ -2z = 2\lambda z &\implies 2z(1+\lambda) = 0 \quad \text{key eq.} & (3)\\ 2x^2 + y^2 + z^2 &= 4 & (4) \end{aligned}$$

From equation (3), we have two possible cases: $z = 0$ or $\lambda = -1$.

Case 1: $z = 0$

From equation (1), either $x = 0$ or $\lambda = \frac{1}{2}$.

- **Subcase 1a:** $x = 0$
  Substituting $x = 0$ and $z = 0$ into (4): $0 + y^2 + 0 = 4 \implies y = \pm 2$.
  This gives the points $(0,2,0)$ and $(0,-2,0)$.
  Evaluating the function: $f(0,2,0) = 4$ and $f(0,-2,0) = -4$. (3p)
- **Subcase 1b:** $\lambda = \frac{1}{2}$
  From (2), $y = \frac{1}{\lambda} = 2$.
  Substituting $y = 2$ and $z = 0$ into (4): $2x^2 + 4 + 0 = 4 \implies 2x^2 = 0 \implies x = 0$.
  This just yields the point $(0,2,0)$ again. (2p)

Case 2: $\lambda = -1$

From (2), $y = \frac{1}{-1} = -1$. (1p)
From (1), $2x(1-2(-1)) = 6x = 0 \implies x = 0$. (1p)
Substituting $x = 0$ and $y = -1$ into (4): $0 + (-1)^2 + z^2 = 4 \implies z^2 = 3 \implies z = \pm\sqrt{3}$. (1p)
This gives the points $(0,-1,\sqrt{3})$ and $(0,-1,-\sqrt{3})$.
Evaluating the function: $f(0,-1,\pm\sqrt{3}) = 0^2 + 2(-1) - (\pm\sqrt{3})^2 = -2 - 3 = -5$ (1p)

Conclusion: (2p)

Comparing the values evaluated at all critical points, the absolute maximum is 4 at the point $(0,2,0)$, and the absolute minimum is -5 at the points $(0,-1,\pm\sqrt{3})$.

Solution 2

(a) Let $\mathbf{F} = \langle P, Q, R \rangle$. We check the curl components:

(5p)

$$\begin{aligned} &P_y = 2xe^y \quad \text{and} \quad Q_x = 2xe^y \implies P_y = Q_x\\ &P_z = \cos(xz) - xz\sin(xz) \quad \text{and} \quad R_x = \cos(xz) - xz\sin(xz) \implies P_z = R_x\\ &Q_z = 0 \quad \text{and} \quad R_y = 0 \implies Q_z = R_y \end{aligned}$$

Since the domain is all of $\mathbb{R}^3$ (which is simply connected) and all cross-partials are equal, $\mathbf{F}$ is conservative.

(b) We need ϕ such that $\phi_x = P$, $\phi_y = Q$, $\phi_z = R$.

$$\phi(x,y,z) = \int (2xe^y + z\cos(xz))\,dx = x^2e^y + \sin(xz) + g(y,z) \quad (1{,}5p)$$

$$\phi_y = x^2e^y + g_y(y,z) = Q = x^2e^y - \sin(y) \implies g_y(y,z) = -\sin(y) \quad (1{,}5p)$$

$$g(y,z) = \cos(y) + h(z) \implies \phi(x,y,z) = x^2e^y + \sin(xz) + \cos(y) + h(z) \quad (1{,}5p)$$

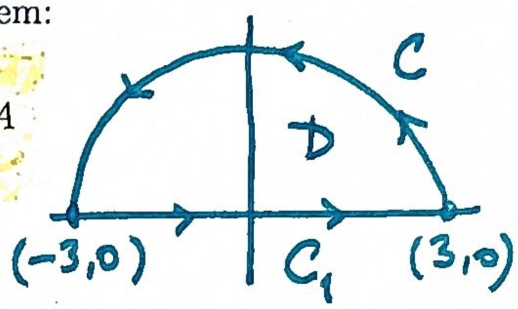
$$\phi_z = x\cos(xz) + h'(z) = R = x\cos(xz) + 2z \implies h'(z) = 2z \implies h(z) = z^2 \quad (1{,}5p)$$

Thus, the potential function is $\phi(x, y, z) = x^2e^y + \sin(xz) + \cos(y) + z^2$. (1.5p)

(c) By the Fundamental Theorem of Line Integrals, $\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(\mathbf{r}(1)) - \phi(\mathbf{r}(0))$.
Start point: $\mathbf{r}(0) = \langle 0, 0, 0\rangle \implies \phi(0, 0, 0) = 0 + 0 + \cos(0) + 0 = 1$.
End point: $\mathbf{r}(1) = \langle 1, 0, \frac{\pi}{2}\rangle \implies \phi(1, 0, \frac{\pi}{2}) = 1^2e^0+\sin(\frac{\pi}{2})+\cos(0)+(\frac{\pi}{2})^2 = 1+1+1+\frac{\pi^2}{4} = 3+\frac{\pi^2}{4}$.
The integral is $(3 + \frac{\pi^2}{4}) - 1 = 2 + \frac{\pi^2}{4}$. (7.5p)

Solution 3

The curve C is the upper half of the circle of radius 3, oriented from $(3, 0)$ to $(-3, 0)$. This is not a closed curve. Let C_1 be the line segment along the x-axis from $(-3, 0)$ to $(3, 0)$.
The combined curve $C \cup C_1$ forms a closed loop K traversing the boundary of the upper half-disk D in the positive (counter-clockwise) direction. By Green's Theorem:



(3p) $$\oint_K P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Here, $P = y^2 + \sin(x^3)$, and $Q = 3xy + e^{y^2}$.

(2p) $$Q_x - P_y = 3y - 2y = y$$

We evaluate the double integral over the upper half-disk D using polar coordinates ($0 \le r \le 3$, $0 \le \theta \le \pi$):

$$\iint_D y\,dA = \int_0^\pi \int_0^3 (r\sin\theta)r\,dr\,d\theta = \left(\int_0^\pi \sin\theta\,d\theta\right)\left(\int_0^3 r^2\,dr\right) = [-\cos\theta]_0^\pi \cdot \left[\frac{r^3}{3}\right]_0^3 = (1-(-1))\cdot 9 = 18$$

So, $\int_C \mathbf{F} \cdot d\mathbf{r} + \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 18$. (6p)
Now we evaluate the line integral over C_1. On C_1, $y = 0 \implies dy = 0$, and x goes from -3 to 3.

$$\int_{C_1} P\,dx + Q\,dy = \int_{-3}^3 (\sin(x^3))\,dx \quad \text{(4p)}$$

Since $f(x) = \sin(x^3)$ is an odd function ($f(-x) = -f(x)$) and the interval $[-3, 3]$ is symmetric around the origin, this integral is strictly 0.
Therefore, $\int_C \mathbf{F} \cdot d\mathbf{r} + 0 = 18 \implies \int_C \mathbf{F} \cdot d\mathbf{r} = 18$.

Solution Key for Part II

Only the final answers are evaluated. No partial credits given.

Solution 4.1

The region of integration is defined by $-8 \le x \le 8$ and $x^{2/3} \le y \le 4$.
The lower bound curve is $y = x^{2/3}$. Solving for x gives $y^3 = x^2$, which means $x = \pm y^{3/2}$.
Since x ranges from -8 to 8, the new horizontal bounds for a fixed y are from the left branch $x = -y^{3/2}$ to the right branch $x = y^{3/2}$.
The total range for y is from 0 (minimum of $x^{2/3}$) to 4.
Answer:

$$\int_0^4 \int_{-y^{3/2}}^{y^{3/2}} f(x,y)\,dx\,dy$$

(3p) (7p)

Solution 4.2

Using the suggested generalized polar coordinates $x = 2r\cos\theta$ and $y = r\sin\theta$.
The Jacobian of this transformation is $J = 2r$.
The boundary of the region D is the elliptic cylinder $x^2+4y^2 = 4$, which simplifies to $4r^2\cos^2\theta + 4r^2\sin^2\theta = 4 \implies r = 1$. Thus, $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.
The volume integral is:

$$\begin{aligned}
V &= \iint_D (x^2+y^2)\,dA \\
&= \int_0^{2\pi}\int_0^1 \left((2r\cos\theta)^2 + (r\sin\theta)^2\right)(2r)\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 2r^3(4\cos^2\theta + \sin^2\theta)\,dr\,d\theta \\
&= \left(\int_0^1 2r^3\,dr\right)\left(\int_0^{2\pi}(4\cos^2\theta+\sin^2\theta)\,d\theta\right) \\
&= \left[\frac{r^4}{2}\right]_0^1 \cdot (4(\pi)+\pi) = \frac{1}{2}\cdot 5\pi = \frac{5\pi}{2}
\end{aligned}$$

Answer: (C) (5p)

Solution 4.3

For the paddle wheel to rotate counter-clockwise, the scalar curl (**k**-component of the curl) must be positive.

$$\begin{aligned}
\operatorname{curl}\mathbf{F}\cdot\mathbf{k} &= \frac{\partial}{\partial x}\left(-\frac{2}{x-2}\right) - \frac{\partial}{\partial y}(y^2) \\
&= \frac{2}{(x-2)^2} - 2y
\end{aligned}$$

Setting the curl greater than zero:

$$\frac{2}{(x-2)^2} - 2y > 0 \implies 2y < \frac{2}{(x-2)^2} \implies y < \frac{1}{(x-2)^2}$$

Since the question asks for the region above the x-axis, we also have $y > 0$.
Answer: (A) (5p)

Solution 4.4
Let $u = x - y$ and $v = x + y$. The bounding lines become $u = 0$, $u = 2$, $v = 0$, and $v = 3$.
Solving for x and y, we get $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(v-u)$.
The absolute value of the Jacobian determinant is $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$.
The integrand becomes $(x+y)e^{x^2-y^2} = ve^{uv}$. The integral transforms to:

$$\begin{aligned}
\int_0^3 \int_0^2 ve^{uv}\left(\frac{1}{2}\right) du\, dv &= \frac{1}{2}\int_0^3 \left[e^{uv}\right]_{u=0}^{u=2} dv \\
&= \frac{1}{2}\int_0^3 (e^{2v} - 1)\, dv \\
&= \frac{1}{2}\left[\frac{1}{2}e^{2v} - v\right]_0^3 = \frac{1}{2}\left(\frac{1}{2}e^6 - 3 - \left(\frac{1}{2} - 0\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{2}e^6 - \frac{7}{2}\right) = \frac{1}{4}(e^6 - 7)
\end{aligned}$$

Answer: (B) (5p)

Solution 4.5
The field lines satisfy the differential equation $\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}$.

$$\frac{dy}{dx} = \frac{x}{y} \implies y\, dy = x\, dx$$

Integrating both sides yields $\frac{1}{2}y^2 = \frac{1}{2}x^2 + K \implies x^2 - y^2 = C$.
Answer: (D) (5p)

Solution 4.6
By the Divergence Theorem, the flux is $\iiint_E \operatorname{div}\mathbf{F}\, dV$.
$\operatorname{div}\mathbf{F} = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$.
Using spherical coordinates for the unit sphere ($0 \le \rho \le 1$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$):

$$\begin{aligned}
\text{Flux} &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (3\rho^2)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= 3\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_0^1 \rho^4\, d\rho\right) \\
&= 3 \cdot (2\pi) \cdot (2) \cdot \left(\frac{1}{5}\right) = \frac{12\pi}{5}
\end{aligned}$$

Answer: (C) (5p)

Solution 4.7
For $\mathbf{F}_1 = \langle -y, x, 0\rangle$: $\operatorname{div}\mathbf{F}_1 = 0 + 0 + 0 = 0$ (Solenoidal). $\operatorname{curl}\mathbf{F}_1 = \langle 0, 0, 2\rangle \neq \mathbf{0}$.
For $\mathbf{F}_2 = \langle x, y, z\rangle$: $\operatorname{div}\mathbf{F}_2 = 1 + 1 + 1 = 3 \neq 0$. $\operatorname{curl}\mathbf{F}_2 = \langle 0, 0, 0\rangle$ (Irrotational).
Answer: (C) (5p)

Solution 4.8
We are looking for $\mathbf{A} = \langle 0, A_2, A_3\rangle$ such that $\nabla \times \mathbf{A} = \langle 2y, z, 0\rangle$.

$$\nabla \times \mathbf{A} = \left\langle \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z},\ -\frac{\partial A_3}{\partial x},\ \frac{\partial A_2}{\partial x}\right\rangle$$

Equating components:

1. $-\frac{\partial A_3}{\partial x} = z \implies A_3 = -xz + g(y,z)$

2. $\frac{\partial A_2}{\partial x} = 0 \implies A_2 = h(y,z)$

3. $\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = 2y$

Substitute A_3 and A_2 into the third equation: $\frac{\partial g}{\partial y} - \frac{\partial h}{\partial z} = 2y$.
There are infinitely many valid choices. A simple choice is to let $h(y,z) = 0$ (so $A_2 = 0$). Then $\frac{\partial g}{\partial y} = 2y \implies g(y,z) = y^2$. This yields $A_3 = y^2 - xz$.
Another valid choice is letting $g(y,z) = 0$ (so $A_3 = -xz$). Then $-\frac{\partial h}{\partial z} = 2y \implies h(y,z) = -2yz$. This yields $A_2 = -2yz$.
Answer: (Using the first choice)

(5p) (5p)

$$\mathbf{A}(x,y,z) = \langle 0\,, 0, y^2 - xz \rangle$$

or any f(y)

(Note: $\langle 0, -2yz, -xz \rangle$ *is also a correct valid pair).*

(5p) (5p)

Other valid answers are also welcome.